# HOMOGENEITY OF THE PURE STATE SPACE OF THE CUNTZ ALGEBRA

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ABSTRACT. If $\omega_1, \omega_2$ are two pure gauge-invariant states of the Cuntz algebra $\mathcal{O}_d$, we show that there is an automorphism $\alpha$ of $\mathcal{O}_d$ such that $\omega_1 = \omega_2 \circ \alpha$. If $\omega$ is a general pure state on $\mathcal{O}_d$ and $\varphi_0$ is a given Cuntz state, we show that there exists an endomorphism $\alpha$ of $\mathcal{O}_d$ such that $\varphi_0 = \omega \circ \alpha$

## 1. INTRODUCTION

Let $\mathfrak{A}$ be a simple separable C*-algebra, and let $\pi_1, \pi_2$ be representations of $\mathfrak{A}$ on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. The representations $\pi_1, \pi_2$ are said to be algebraically equivalent if $\pi_1(\mathfrak{A})''$ and $\pi_2(\mathfrak{A})''$ are isomorphic von Neumann algebras. If there is an automorphism $\alpha$ of $\mathfrak{A}$ such that $\pi_1$ and $\pi_2 \circ \alpha$ are quasi-equivalent, then $\pi_1, \pi_2$ are clearly algebraically equivalent. Powers proved in [Pow67] that if $\mathfrak{A}$ is a UHF algebra the converse is true. His method extends readily to the case that $\mathfrak{A}$ is an AF-algebra, [Bra72]. See also section 12.3 in [KR86]. In the special case that $\pi_1$ (and therefore $\pi_2$) is irreducible, Kadison's transitivity theorem therefore implies that if $\mathfrak{A}$ is a simple AF algebra and if $\omega_1$ and $\omega_2$ are pure states on $\mathfrak{A}$, there exists an automorphism $\alpha$ of $\mathfrak{A}$ such that $\omega_1 = \omega_2 \circ \alpha$. To our knowledge, this question has only been settled in the affirmative when $\mathfrak{A}$ is an AF-algebra. As a beginning of a possible resolution of the question for purely infinite algebras, we here prove the statements in the abstract. Recall from [Cun77] that the Cuntz algebra $\mathcal{O}_d$ is the C*-algebra generated by $d$ operators $s_1, \dots, s_d$ satisfying

$$s_j^* s_i = \delta_{ij} \mathbb{1}$$
$$\sum_{i=1}^{d} s_i s_i^* = \mathbb{1}$$

There is an action $\gamma$ of the group $U(d)$ of unitary $d \times d$ matrices on $\mathcal{O}_d$ given by

$$\gamma_g(s_i) = \sum_{j=1}^{d} g_{ji} s_j$$

for $g = [g_{ij}]_{i,j=1}^d$ in $U(d)$. In particular the *gauge action* $\tau = \gamma|_{\mathbf{T}}$ is defined by

$$\tau_z(s_i) = z s_i \ , \qquad z \in \mathbf{T} \subset \mathbf{C} \ .$$

If $\mathrm{UHF}_d$ is the fixed point subalgebra under the gauge action, then $\mathrm{UHF}_d$ is the closure of the linear span of all Wick ordered polynomials of the form

$$s_{i_1} \dots s_{i_k} s_{j_k}^* \dots s_{j_1}^*$$

$\mathrm{UHF}_d$ is isomorphic to the UHF algebra of Glimm type $d^\infty$:

$$\mathrm{UHF}_d \cong M_{d^\infty} = \bigotimes_1^\infty M_d$$

in such a way that the isomorphism carries the Wick ordered polynomial above into the matrix element

$$e^{(1)}_{i_1 j_1} \otimes e^{(2)}_{i_2 j_2} \otimes \cdots \otimes e^{(k)}_{i_k j_k} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots .$$

In the case that $d$ is a power of a prime, the gauge action $\tau$ is in fact characterized by the fact that its fixed point algebra is isomorphic to $\mathrm{UHF}_d$, i.e. if $\alpha$ is another faithful action of $\mathbf{T}$ on $\mathcal{O}_d$ such that the fixed point algebra $\mathcal{O}_d^\alpha$ is isomorphic to $\mathrm{UHF}_d$, then either $z \mapsto \alpha_z$ or $z \mapsto \alpha_z^{-1}$ is conjugate to $\tau$. This follows from [BK99, Corollary 4.1]. (Since $\mathrm{UHF}_d$ is simple and $\alpha$ is faithful, the crossed product $\mathcal{O}_d \times_\alpha \mathbf{T}$ is stably isomorphic to $\mathrm{UHF}_d$, [KT78], and in particular it is simple. Since

$$\mathcal{O}_d^\alpha \cong P_\alpha(0)(\mathcal{O}_d \times_\alpha \mathbf{T})P_\alpha(0) ,$$

$[P_\alpha(0)]$ is just $[\mathbb{1}]$ when $K_0(\mathcal{O}_d \times_\alpha \mathbf{T})$ is identified with $K_0(\mathcal{O}_d^\alpha)$. By the Pimsner-Voiculescu exact sequence it follows that $\widehat{\alpha}_*$ on $K_0(\mathcal{O}_d \times_\alpha \mathbf{T}) = \mathbf{Z}[\frac{1}{d}]$ is multiplication by $d$ or $1/d$. For this last argument it is important that $d$ is a power of a prime, as seen from the example $d = 6$ and $\widehat{\alpha}_*$ equal to multiplication by $4/9$ on $\mathbf{Z}[\frac{1}{6}]$) Because of this, our main result Theorem 5 can be given the following more universal form:

**Corollary 1.** *Assume that $d$ is a power of a prime. Let $\varphi_1$ and $\varphi_2$ be pure states on $\mathcal{O}_d$, and assume that there exist actions $\alpha_i$ of $\mathbf{T}$ on $\mathcal{O}_d$ such that $\mathcal{O}_d^{\alpha_i} \cong \mathrm{UHF}_d$ and $\varphi_i \circ \alpha_i = \varphi_i$ for $i = 1, 2$. Then there exists an automorphism $\beta$ of $\mathcal{O}_d$ such that*

$$\varphi_1 = \varphi_2 \circ \beta$$

The question whether any pure state on $\mathcal{O}_d$ is invariant under a gauge action is left open.

The restriction of $\gamma_g$ to $\mathrm{UHF}_d$ is carried into the action

$$\mathrm{Ad}(g) \otimes \mathrm{Ad}(g) \otimes \cdots$$

on $\bigotimes_1^\infty M_d$. We define the canonical endomorphism $\lambda$ on $\mathrm{UHF}_d$ (or on $\mathcal{O}_d$) by

$$\lambda(x) = \sum_{j=1}^d s_j x s_j^*$$

and the isomorphism carries $\lambda$ over into the one-sided shift

$$x_1 \otimes x_2 \otimes x_3 \otimes \cdots \to \mathbb{1} \otimes x_1 \otimes x_2 \otimes \cdots$$

on $\bigotimes_1^\infty M_d$.

If $\eta_1, \dots, \eta_d$ are complex scalars with $\sum_{j=1}^d |\eta_j|^2 = 1$, we can define a state on $\mathcal{O}_d$ by

$$\varphi_\eta(s_{i_1} \dots s_{i_k}\ s_{j_\ell}^* \dots s_{j_1}^*) = \eta_{i_1} \dots \eta_{i_k}\ \overline{\eta_{j_\ell}} \dots \overline{\eta_{j_1}}$$

[Cun77], [Eva80], [BJP96], [BJ97], [BJKW].

This state is pure, and non-gauge invariant, and the $U(d)$ action is transitive on these states, which are called Cuntz states. The restriction of $\varphi_\eta$ to $\mathrm{UHF}_d$

identifies with the pure product state given by infinitely many copies of the vector state defined by the vector $(\eta_1, \ldots, \eta_d)$ on $M_d$.

In this paper we will also consider the one-one correspondence between the set $\mathcal{U}(\mathcal{O}_d)$ of unitaries in $\mathcal{O}_d$ and the set $\mathrm{End}(\mathcal{O}_d)$ of unital endomorphisms of $\mathcal{O}_d$. If $u \in \mathcal{U}(\mathcal{O}_d)$ then $\alpha_u(s_i) = us_i$ defines an endomorphism, and if $\alpha \in \mathrm{End}(\mathcal{O}_d)$ the corresponding unitary is $u = \sum_{i=1}^{d} \alpha(s_i)s_i^*$. It has been proved by Rørdam that

$$\mathcal{U}_i = \{u \in \mathcal{U}(\mathcal{O}_d) | \alpha_u \text{ is an inner automorphism}\}$$

is a dense subset of $\mathcal{U}(\mathcal{O}_d)$, [Rør93]. We give a shorter proof of this, and also show that

$$\mathcal{U}_a = \{u \in \mathcal{U}(\mathcal{O}_d) | \alpha_u \text{ is an automorphism}\}$$

is a dense $G_\delta$ subset of $\mathcal{U}(\mathcal{O}_d)$ such that the complement $\mathcal{U}(\mathcal{O}_d) \setminus \mathcal{U}_a$ is also dense.

By using the above correspondence between $\mathcal{U}(\mathcal{O}_d)$ and $\mathrm{End}(\mathcal{O}_d)$, it follows (see the proof of Proposition 8) that if $\omega$ is a pure state and $\varphi_0$ a Cuntz state there exists an endomorphism $\alpha$ of $\mathcal{O}_d$ such that $\varphi_0 = \omega \circ \alpha$. Although the automorphism group is dense in $\mathrm{End}(\mathcal{O}_d)$ (in the topology of pointwise convergence), the question whether $\alpha$ can be chosen to be an automorphism is left open (in this approach).

## 2. Transitivity of the automorphism group on the pure gauge-invariant states

In this section we prove the first main result mentioned in the abstract.

Let $\mathrm{UHF}_d$ be the UHF algebra of type $d^\infty$ and let $(A_n)$ be an increasing sequence of C*-subalgebras of $\mathrm{UHF}_d$ such that $\mathrm{UHF}_d = \overline{\cup A_n}$ and $A_n \cong M_{d^n}$. We first use Power's transitivity on $\mathrm{UHF}_d$ to find an approximate factorization for any pure state on $\mathrm{UHF}_d$:

**Lemma 2.** *Let $\varphi$ be a pure state of $\mathrm{UHF}_d$ and $\varepsilon > 0$. Then there exists a pure state $\varphi'$ of $\mathrm{UHF}_d$, an increasing sequence $\{B_n\}$ of finite type I subfactors of $\mathrm{UHF}_d$, and an increasing subsequence $\{k_n\}$ in $\mathbf{N}$ such that $\varphi'|B_n$ is a pure state of $B_n$ and $A_{k_n} \subset B_n \subset A_{k_{n+1}}$ for every $n$, and*

$$\|\varphi - \varphi'\| < \varepsilon\ .$$

*Proof.* Since the automorphism group $\mathrm{Aut}(\mathrm{UHF}_d)$ of $\mathrm{UHF}_d$ acts transitively on the set of pure states of $\mathrm{UHF}_d$, [Pow67], there exists an increasing sequence $\{D_n\}$ of finite type I subfactors of $\mathrm{UHF}_d$ such that $D_n \cong M_{d^n}$ and $\varphi|D_n$ is pure for every $n$. Then we can find sequences $\{u_n\}$ and $\{v_n\}$ of unitaries in $\mathrm{UHF}_d$ and increasing sequences $\{k_n\}$ and $\{\ell_n\}$ in $\mathbf{N}$ such that

$$\begin{aligned}
&A_{k_1} \subset \mathrm{Ad}(v_1u_1)(D_{\ell_1}) \subset A_{k_2} \subset \mathrm{Ad}(v_2u_2v_1u_1)(D_{\ell_2}) \subset A_{k_3} \subset \cdots \\
&u_n \in \mathrm{UHF}_d \cap \mathrm{Ad}(v_{n-1}u_{n-1} \ldots v_1u_1)(D_{\ell_{n-1}})' \\
&v_n \in \mathrm{UHF}_d \cap A_{k_n}' \\
&\|u_n - 1\| < \varepsilon/2^{n+2} \qquad \|v_n - 1\| < \varepsilon/2^{n+2}
\end{aligned}$$

where $D_0 = \mathbf{C}1$. (Let $k_1 = 1$. Then we choose $u_1$ and $\ell_1$ such that $A_{k_1} \subset \mathrm{Ad}\, u_1(D_{\ell_1})$ and $\|u_1 - 1\| < \varepsilon/8$. Further we choose $k_2$ and $v_1$ such that $v_1 \in \mathrm{UHF}_d \cap A_{k_1}'$, $\|v_1 - 1\| < \varepsilon/8$, and, $\mathrm{Ad}(v_1u_1)(D_{\ell_1}) \subset A_{k_2}$. We just repeat this

process.) Then the limit $w = \lim v_n u_n \dots v_1 u_1$ exists and is a unitary such that $\|w - 1\| < \varepsilon/2$ and

$$A_{k_1} \subset \operatorname{Ad} w(D_{\ell_1}) \subset A_{k_2} \subset \operatorname{Ad} w(D_{\ell_2}) \subset \cdots$$

Let $\varphi' = \varphi \circ \operatorname{Ad} w^*$. Then $\varphi'$ is a pure state with $\|\varphi - \varphi'\| < \varepsilon$ and $\varphi' | \operatorname{Ad} w(D_{\ell_n})$ is a pure state for every $n$. Put $B_n = \operatorname{Ad} w(D_{\ell_n})$. □

We next show that for any pair of pure states $\varphi_1, \varphi_2$ on $\mathrm{UHF}_d$, there is a tensor product decomposition of $\mathrm{UHF}_d$ such that $\varphi_1, \varphi_2$ have approximate factorizations with respect to certain sub-decompositions (necessarily different for $\varphi_1$ and $\varphi_2$):

**Lemma 3.** *Let $\varphi_1$ and $\varphi_2$ be pure states of $\mathrm{UHF}_d$ and let $\varepsilon > 0$. Then there exist pure states $\varphi_1', \varphi_2'$, and $\psi$ of $\mathrm{UHF}_d$, an increasing sequence $\{k_n\}$ in $\mathbf{N}$ and an increasing sequence $\{B_n\}$ of finite type I subfactors of $A$ such that*

$$\begin{aligned}
&\|\varphi_i - \varphi_i'\| < \varepsilon \\
&\varphi_1' | B_{2n+1} \quad \textit{is pure} \\
&\varphi_2' | B_{2n} \quad \textit{is pure} \\
&\psi | B_{6k-1} \cap B_{6k-3}' = \varphi_1' | B_{6k-1} \cap B_{6k-3}' \\
&\psi | B_{6k+2} \cap B_{6k}' = \varphi_2' | B_{6k+2} \cap B_{6k}' \\
&\psi | B_{6k} \cap B_{6k-1}' \quad \textit{is pure,} \\
&\psi | B_{6k-3} \cap B_{6k-4}' \quad \textit{is pure,} \\
&k_{n+1} - k_n \to \infty \\
&A_{k_1} \subset B_1 \subset A_{k_2} \subset B_2 \subset A_{k_3} \subset B_3 \subset \cdots
\end{aligned}$$

*Proof.* It follows from the previous lemma that there exist pure states $\varphi_i'$, increasing sequences $\{B_{in}\}$ of finite type I subfactors of $A$, and an increasing sequence $\{k_n\}$ in $\mathbf{N}$ such that

$$\begin{aligned}
&\|\varphi_i - \varphi_i'\| < \varepsilon \, , \\
&\varphi_i | B_{in} \quad \text{is pure for } i = 1, 2 \, , \\
&A_{k_1} \subset B_{i1} \subset A_{k_2} \subset B_{i2} \subset A_{k_3} \subset \cdots
\end{aligned}$$

By passing to subsequences of $\{k_n\}$ and $\{B_{in}\}$ and setting $B_n = B_{1n}$ if $n$ is odd and $B_n = B_{2n}$ if $n$ is even, we may assume that

$$\begin{aligned}
&\varphi_1' | B_{2n+1} \quad \text{is pure} \\
&\varphi_2' | B_{2n} \quad \text{is pure} \\
&k_{n+1} - k_n \to \infty \\
&A_{k_1} \subset B_1 \subset A_{k_2} \subset B_2 \subset A_{k_3} \subset \cdots
\end{aligned}$$

Then $\varphi_1'$ has a tensor product decomposition into pure states on the matrix subalgebras $B_{2n+1} \cap B_{2n-1}'$, and $\varphi_2'$ likewise on the subalgebras $B_{2n} \cap B_{2n-2}'$. Thus we can define a pure state $\psi$ by requiring that it decomposes under the tensor product decomposition

$$\begin{aligned}
\dots \otimes (B_{6k-4} \cap B_{6k-6}') \otimes (B_{6k-3} \cap B_{6k-4}') \otimes (B_{6k-1} \cap B_{6k-3}') \\
\otimes (B_{6k} \cap B_{6k-1}') \otimes (B_{6k+2} \cap B_{6k}') \otimes \cdots
\end{aligned}$$

into states given by:

$$\begin{aligned}
&\psi|B_{6k-1}\cap B'_{6k-3} = \varphi'_1|B_{6k-1}\cap B'_{6k-3}\ ,\\
&\psi|B_{6k+2}\cap B'_{6k} = \varphi'_2|B_{6k+2}\cap B'_{6k}\ ,\\
&\psi|B_{6k}\cap B'_{6k-1} \quad \text{is an arbitrary pure state,}\\
&\psi|B_{6k-3}\cap B'_{6k-4} \quad \text{is an arbitrary pure state.}
\end{aligned}$$

□

Recall that $\tau$ is the gauge action of $\mathbf{T}$ on $\mathcal{O}_d$, i.e.,

$$\tau_z(s_i) = zs_i\ , \qquad z\in\mathbf{T}\ .$$

Let $\varepsilon$ be the conditional expectation of $\mathcal{O}_d$ onto $\mathrm{UHF}_d$ defined by

$$\varepsilon(x) = \int_{\mathbf{T}} \tau_z(x)\frac{|dz|}{2\pi}\ , \qquad x\in\mathcal{O}_d\ .$$

Note that if $\varphi$ is a gauge-invariant state of $\mathcal{O}_d$, then

$$\varphi = \varphi|_{\mathrm{UHF}_d}\circ\varepsilon\ .$$

Recall that $\lambda$ is canonical endomorphism of $\mathcal{O}_d$: $\lambda(x) = \sum_{i=1}^{d} s_i x s_i^*$, $x\in\mathcal{O}_d$, and that the restriction of $\lambda$ to $\mathrm{UHF}_d$ is the one-sided shift $\sigma$.

**Lemma 4.** *If $\varphi$ is a gauge-invariant state on $\mathcal{O}_d$ then the following conditions are equivalent:*

(i) *$\varphi$ is pure*

(ii) *$\varphi|_{\mathrm{UHF}_d}$ is pure and*
*$\varphi|_{\mathrm{UHF}_d}\circ\sigma^n$ is disjoint from $\varphi$ for $n = 1, 2, \ldots$*

*Proof.* (i)⇒(ii). Since $\varphi$ is pure, and gauge-invariant, it follows that $\varphi|_{\mathrm{UHF}_d}$ is pure. Let $p$ be the support projection of $\varphi$ in $\mathcal{O}_d^{**}$. Since $p$ is minimal, and $\varphi$ is gauge-invariant, it follows that for any $a\in\mathrm{UHF}_d$ and any multi-index $I = (i_1, i_2, \ldots, i_n)$ with $|I| = n \geq 1$,

$$pas_Ip = \varphi(as_I)p = 0\ ,$$

where $s_I = s_{i_1}s_{i_2}\ldots s_{i_n}$. Thus we obtain that

$$p(\mathrm{UHF}_d)\lambda^n(p) = 0\ ,$$

which implies that $\varphi|_{\mathrm{UHF}_d}\circ\sigma^n$ is disjoint from $\varphi$.

(ii)⇒(i). Let $p$ be the support projection of $\varphi|_{\mathrm{UHF}_d}$ in $\mathrm{UHF}_d^{**}\subset\mathcal{O}_d^{**}$. It suffices to show that for any multi-indices $I, J$

$$ps_Is_J^*p\in\mathbf{C}p$$

since the linear span of $s_Is_J^*$ is dense in $\mathcal{O}_d$. If $|I|\neq[J|$, we have that $ps_Is_J^*p = 0$ by using the fact that $\varphi|_{\mathrm{UHF}_d}\circ\sigma^n$ is disjoint from $\varphi$ for $n = \Big||I|-|J|\Big|$. If $[I| = |J|$, we have that $ps_Is_J^*p = \varphi(s_Is_J^*)p$ since $\varphi|_{\mathrm{UHF}_d}$ is pure. □

**Lemma 5.** *Let $\varphi_1$ and $\varphi_2$ be gauge-invariant pure states of $\mathcal{O}_d$ such that all $\varphi_i|_{\mathrm{UHF}_d}\circ\sigma^n$, $i = 1, 2$, $n = 0, 1, 2, \ldots$ are mutually disjoint. Then there exists an automorphism $\alpha$ of $\mathcal{O}_d$ such that $\alpha\circ\tau_z = \tau_z\circ\alpha$, $z\in\mathbf{T}$ and $\varphi_1 = \varphi_2\circ\alpha$.*

*Proof.* By Lemma 4, $\psi_1 = \varphi_1|_{\mathrm{UHF}_d}$ and $\psi_2 = \varphi_2|_{\mathrm{UHF}_d}$ are pure states on $\mathrm{UHF}_d$. Applying Lemma 3 on $\psi_1, \psi_2$ in lieu of $\varphi_1, \varphi_2$, with $\varepsilon = 1$, we obtain pure states $\psi_1', \psi_2'$ and $\psi$ of $\mathrm{UHF}_d$ with the properties given there. Since $\psi_i$ is equivalent to $\psi_i'$, $\varphi_i' = \psi_i' \circ \varepsilon$ is a pure state of $\mathcal{O}_d$ by Lemma 4 and this state is equivalent to $\varphi_i = \psi_i \circ \varepsilon$. By Kadison's transitivity theorem we have a unitary $u \in \mathrm{UHF}_d$ such that $\psi_i' = \psi_i \circ \mathrm{Ad}\, u$; it follows that $\varphi_i' = \varphi_i \circ \mathrm{Ad}\, u$.

It is not automatical that $\psi$ satisfies the condition that all $\psi \circ \sigma^n$, $n = 0, 1, 2, \dots$ are mutually disjoint and are disjoint from $\psi_i' \circ \sigma^n$. But using the freedom in constructing $\psi|_{B_{6k} \cap B_{6k-1}'}$ and $\psi|_{B_{6k-3} \cap B_{6k-4}'}$ successively, we can certainly impose this condition.

Thus we obtain three pure states $\psi_1', \psi_2', \psi$ of $\mathrm{UHF}_d$ such that all $\psi_i' \circ \sigma^n$, $\psi \circ \sigma^n$ are mutually disjoint and $\psi_i'$ and $\psi$ are spotwise asymptotically equal as specified in Lemma 3. It now suffices to prove the lemma for the pairs $(\psi_1' \circ \varepsilon, \psi \circ \varepsilon)$ and $(\psi_2' \circ \varepsilon, \psi \circ \varepsilon)$. Thus replacing $\varphi_1, \varphi_2$ by one of these pairs, we may assume the lemma satisfy the additional condition that there exists an increasing sequence $\{k_n\}$ in $\mathbf{N}$ and an increasing sequence $\{B_n\}$ of finite type I subfactors of $\mathrm{UHF}_d$ such that

$$\begin{aligned} & A_{k_1} \subset B_1 \subset A_{k_2} \subset B_2 \subset A_{k_3} \subset B_3 \subset \\ & \varphi_i|_{B_{3n+1}} \quad \text{is pure ,} \\ & \varphi_1|_{B_{3n+3} \cap B_{3n+1}'} = \varphi_2|_{B_{3n+3} \cap B_{3n+1}'} \quad \text{is pure} \\ & k_{3n+3} - k_{3n+2} \to \infty \ . \end{aligned}$$

We shall construct a sequence $\{v_n\}$ of unitaries in $\mathrm{UHF}_d$ such that $\alpha = \lim_{n \to \infty} \mathrm{Ad}(v_n v_{n-1} \dots v_1)$ defines an automorphism of $\mathcal{O}_d$ with $\varphi_1 = \varphi_2 \circ \alpha$. To ensure the existence of the limit we choose the unitaries such that they mutually commute and $\sum \|\lambda(v_n) - v_n\| < \infty$. Since $\alpha$ commutes with the gauge action $\tau$, this will complete the proof.

We fix a large $N \in \mathbf{N}$. We choose $n_1$ so large that the support projections $e_i^{(1)} = \mathrm{supp}(\varphi_i|_{B_{3n_1+1}})$ are almost orthogonal and $k_{3n_1+3} - k_{3n_1+2} > 2^{2(N+1)}$. Let $w_1$ be a partial isometry in $B_{3n_1+1}$ with $w_1^* w_1 = e_1^{(1)}$, $w_1 w_1^* = e_2^{(1)}$. By the polar decomposition of the approximate unitary

$$w_1 + (1 - e_2^{(1)}) w_1^* (1 - e_1^{(1)}) + (1 - e_2^{(1)})(1 - e_1^{(1)}) \ ,$$

we obtain a unitary $v_1 \in B_{3n_1+1}$ such that

$$v_1 e_1^{(1)} = w_1 e_1^{(1)} = e_2^{(1)} w_1 = e_2^{(1)} v_1 \in B_{3n_1+1}$$

and $v_1(1 - e_2^{(1)})(1 - e_1^{(1)}) \approx (1 - e_2^{(1)})(1 - e_1^{(1)})$.

We next choose $n_2 > n_1$ so large that

$$\sigma^n \circ \mathrm{supp}(\varphi_i|_{B_{3n_2+1} \cap B_{3n_1+3}'}), \quad i = 1, 2, \ \ n = -2^{N-1}, -2^{-N+1} + 1, \dots, 0, \dots, 2^{N+1}$$

are almost orthogonal and $k_{3n_2+2} - k_{3n_1+1} > 2^{2(N+2)}$. (Though $\sigma$ is an endomorphism, $\sigma^{-n}$ on $B_{3n_2+1} \cap B_{3n_1+3}'$ is well defined for $n = 1, 2, \dots, k_{3n_1+2}$.) Let $w_2$ be a partial isometry in $B_{3n_2+1} \cap B_{3n_2+3}'$ such that

$$w_2^* w_2 = e_1^{(2)} = \mathrm{supp}(\varphi_1|_{B_{3n_2+1} \cap B_{3n_1+3}'})$$

and

$$w_2 w_2^* = e_2^{(2)} = \mathrm{supp}(\varphi_2|_{B_{3n_2+1} \cap B_{3n_1+3}'}) \ ,$$

and let $\zeta$ be a partial isometry in $A_{k_{3n_2+2}+1} \cap A'_{k_{3n_1+3}}$ such that $\zeta^*\zeta = e_1^{(2)}$ and $\zeta\zeta^* = \sigma(e_1^{(2)})$.

Assume for the moment that $\sigma^\ell(e_i^{(2)})$, $i = 1, 2$; $\ell = -2^{N+1}, -2^{N+1}+1, \ldots, 2^{N+1}$ are all orthogonal and set

$$e_{ij} = \begin{cases} \sigma^{i-1}(\zeta)\sigma^{i-2}(\zeta)\ldots\sigma^{j}(\zeta) & i > j \\ \sigma^i(e_1^{(2)}) & i = j \\ \sigma^i(\zeta^*)\sigma^{i+1}(\zeta^*)\ldots\sigma^{j-1}(\zeta^*) & i < j \end{cases}$$

for $i, j = -2^{-N+1}, \ldots, 2^{N+1}$. Then $(e_{ij})$ is a family of matrix units such that $\sigma(e_{ij}) = e_{i+1,j+1}$ when $|i|, |i+1|, |j|, |j+1| \leq 2^{N+1}$. Let

$$\begin{aligned} E = e_1^{(2)} + \sum_{\ell=1}^{2^{N+1}-1} (1 - e_1^{(2)}) \Big\{ & \frac{2^{N+1} - \ell}{2^{N+1}} e_{\ell,\ell} + \frac{\ell}{2^{N+1}} e_{\ell - 2^{N+1}, \ell - 2^{N+1}} \\ & + \frac{1}{2^{N+1}} \sqrt{(2^{N+1} - \ell)\ell}\, (e_{\ell, \ell - 2^{-N+1}} + e_{\ell - 2^{-N+1}, \ell}) \Big\} (1 - e_1^{(2)}) \end{aligned}$$

as in [Kis95]. Then $E$ is a projection in $D_2 = A_{(k_{3n_2+2}+2^{N+1})} \cap A'_{(k_{3n_1+3}-2^{N+1})}$ and satisfies

$$\|\sigma(E) - E\| \sim \frac{1}{2^{\frac{N+1}{2}}} .$$

Let $w = w_2 + (1 - e_2^{(2)})\Big(\sum_{\ell=1}^{2^{N+1}} (\sigma^\ell(w_2) + \sigma^{-\ell}(w_2))\Big)(1 - e_1^{(2)})$ and

$$v = wE + (1 - F)w^*(1 - E) + (1 - F)(1 - E)$$

where $F = wEw^*$.

By the orthogonality assumption on $\sigma^\ell(e_i^{(2)})$, $v$ is a unitary in $D_2$ and satisfies

$$\begin{aligned} &\|\sigma(v) - v\| \approx \|\sigma(E) - E\| , \\ &v e_1^{(2)} = w_2 e_1^{(2)} = e_2^{(2)} w_2 = e_2^{(2)} v . \end{aligned}$$

Note also that $v$ commutes with $v_1$ and $e_i^{(1)}$.

Now, the projections $\sigma^\ell(e_i^{(2)})$, $i = 1, 2$, $\ell = -2^{N+1}, \ldots, 2^{N+1}$ are not actually orthogonal but choosing $n_2$ so large that they are very close to being orthogonal, we may obtain a unitary $v_2$ in $D_2$ by polar decomposition of $v$ such that $v_2$ satisfies the same conditions as above, i.e.,

$$\begin{aligned} &v_2 e_1^{(2)} = w_2 e_1^{(2)} = e_2^{(2)} w_2 = e_2^{(2)} v_2 \in B_{3n_2+1} \cap B'_{3n_1+3} , \\ &\|\lambda(v_2) - v_2\| \sim 2^{-\frac{N+1}{2}} \end{aligned}$$

and $v_2 \in D_2$.

Since

$$\begin{aligned} &\mathrm{supp}(\varphi_1|_{B_{3n_2+1}}) \\ &\quad = \mathrm{supp}(\varphi_1|_{B_{3n_1+1}})\,\mathrm{supp}(\varphi_1|_{B_{3n_1+3}\cap B'_{3n_1+1}})\,\mathrm{supp}(\varphi_1|_{B_{3n_2+1}\cap B'_{3n_1+3}}) \\ &\quad = e_1^{(1)} p e_1^{(2)} \end{aligned}$$

with $p = \operatorname{supp}(\varphi_1|_{B_{3n_1+3}\cap B'_{3n_1+1}}) = \operatorname{supp}(\varphi_2|_{B_{3n_1+3}\cap B'_{3n_1+1}})$, and since the operators $v_1e_1^{(1)} = e_2^{(1)}v_1, p$, and $v_2e_1^{(2)} = e_2^{(2)}v_2$ commute, we obtain that

$$\begin{aligned} v_1v_2 \cdot \operatorname{supp}(\varphi_1|_{B_{3n_2+1}}) &= v_1v_2e_1^{(1)}pe_1^{(2)} \\ &= v_1e_1^{(1)}v_2e_1^{(2)}p \\ &= e_2^{(1)}v_1e_2^{(2)}v_2p \\ &= pe_2^{(1)}e_2^{(2)}v_1v_2 = \operatorname{supp}(\varphi_2|_{B_{3n_2+1}})v_1v_2 \ . \end{aligned}$$

Here we have also used the fact that $v_1$ commutes with $e_2^{(2)}$. We repeat this procedure. Thus we obtain an increasing sequence $\{n_k\}$ in $\mathbf{N}$ and a sequence $\{v_k\}$ of mutually commuting unitaries such that

$$\begin{aligned} &\|\lambda(v_k) - v_k\| \sim 2^{-\frac{N+k}{2}} \ , \\ &v_ke_1^{(k)} = e_2^{(k)}v_k \in \mathcal{B}_{3n_k+1} \cap \mathcal{B}'_{3n_{k-1}+3} \end{aligned}$$

where

$$e_i^{(k)} = \operatorname{supp}(\varphi_i|_{\mathcal{B}_{3n_k+1}\cap\mathcal{B}'_{3n_{k-1}+3}}) \ ,$$

and such that $\operatorname{Ad}(v_k \dots v_1)$ maps $\operatorname{supp}(\varphi_1|_{\mathcal{B}_{3n_k+1}})$ into $\operatorname{supp}(\varphi_2|_{\mathcal{B}_{3n_k+1}})$. Then the limit $\alpha = \lim_k \operatorname{Ad}(v_k \dots v_1)$ defines the desired automorphism. $\square$

**Theorem 6.** *Let $\varphi_1$ and $\varphi_2$ be gauge-invariant pure states of $\mathcal{O}_d$. Then there exists an automorphism $\alpha$ of $\mathcal{O}_d$ such that $\varphi_1 = \varphi_2 \circ \alpha$.*

*Proof.* If $\varphi_1$ is disjoint from $\varphi_2$, then it follows that $(\varphi_i|_{\mathrm{UHF}_d}) \circ \sigma^n = \varphi_i \circ \lambda^n|_{\mathrm{UHF}_d}$, $i = 1, 2$, $n = 0, 1, 2, \dots$ are mutually disjoint (by Lemma 4); thus the assertion follows from Lemma 5. If $\varphi_1$ is equivalent to $\varphi_2$, there is a unitary $u \in \mathcal{O}_d$ such that $\varphi_1 = \varphi_2 \operatorname{Ad} u$ (by Kadison's transitivity). $\square$

## 3. Pure states mapped into Cuntz states by endomorphisms

There is a one-to-one correspondence between the set $\mathcal{U}(\mathcal{O}_d)$ of unitaries of $\mathcal{O}_d$ and the set $\operatorname{End}(\mathcal{O}_d)$ of unital endomorphisms of $\mathcal{O}_d$; if $u \in \mathcal{U}(\mathcal{O}_d)$, the endomorphism $\alpha_u$ is defined by $\alpha_u(s_i) = us_i$ and if $\alpha \in \operatorname{End}(\mathcal{O}_d)$, $\alpha$ corresponds to the unitary $u$ defined by $u = \sum_{i=1}^{d} \alpha(s_i)s_i^*$. Define

$$\begin{aligned} \mathcal{U}_i &= \{u \in \mathcal{U}(\mathcal{O}_d) | \quad \alpha_u \ \text{ is an inner automorphism}\} \\ \mathcal{U}_a &= \{u \in \mathcal{U}(\mathcal{O}_d) | \quad \alpha_u \ \text{ is an automorphism}\} \\ \mathcal{U}_s &= \mathcal{U}(\mathcal{O}_d) \setminus \mathcal{U}_a \ . \end{aligned}$$

**Proposition 7.** *Let $\mathcal{U}_i, \mathcal{U}_a, \mathcal{U}_s$ be as above.*
 *(i) $\mathcal{U}_i$ is a dense subset of $\mathcal{U}(\mathcal{O}_d)$.*
 *(ii) $\mathcal{U}_a$ is a dense $G_\delta$ subset of $\mathcal{U}(\mathcal{O}_d)$.*
 *(iii) $\mathcal{U}_s$ is a dense $F_\sigma$ subset of $\mathcal{U}(\mathcal{O}_d)$.*

*Proof.* M. Rørdam proved (i) in [Rør93] and the other statements are more or less known.

We shall give a proof of (i). We again denote by $\lambda$ the canonical endomorphism of $\mathcal{O}_d : \lambda(x) = \sum_{i=1}^{d} s_ixs_i^*$, $x \in \mathcal{O}_d$. Since the unitary corresponding to $\operatorname{Ad} v$ is $v\lambda(v^*)$,

it suffices to show that $v\lambda(v^*)$, $v \in \mathcal{U}(\mathcal{O}_d)$, is dense in $\mathcal{U}(\mathcal{O}_d)$. If $\mathrm{UHF}_d$ denotes the C*-subalgebra generated by $s_{i_1}s_{i_2}\dots s_{i_n}s_{j_n}^*\dots s_{j_1}^*$, then we mentioned in the introduction that $\mathrm{UHF}_d$ is isomorphic to the UHF algebra $\bigotimes_{\mathbf{N}} M_d$ and $\lambda|\mathrm{UHF}_d$ corresponds to the one-sided shift on $\bigotimes_{\mathbf{N}} M_d$. Thus $\lambda|\mathrm{UHF}_d$ satisfies the Rohlin property, [BKRS93], [Kis95]. In particular for any $n$ and $\varepsilon > 0$ there is an orthogonal family $e_0, e_1, \dots, e_{n-1}$ of projections in $\mathrm{UHF}_d$ such that

$$\sum_{i=0}^{d^n-1} e_i = 1$$
$$\|\lambda(e_i) - e_{i+1}\| < \varepsilon$$

with $e_{d^n} = e_0$. The similar properties hold for $\mathrm{Ad}\,u \circ \lambda$, i.e., if $\mathrm{UHF}_d^u$ denotes the C*-subalgebra generated by $us_{i_1}us_{i_2}\dots us_{i_n}s_{j_n}^*u^*\dots s_{j_1}^*u^*$, then $\mathrm{Ad}\,u \circ \lambda|\mathrm{UHF}_d^u$ corresponds to the one-sided shift on $\bigotimes_{\mathbf{N}} M_d$. Hence for any $n$ and $\varepsilon > 0$ there is an orthogonal family $f_0, f_1, \dots, f_{d^n-1}$ of projections in $\mathrm{UHF}_d^u$ such that

$$\sum_{i=0}^{d^n-1} f_i = 1$$
$$\|\mathrm{Ad}\,u \circ \lambda(f_i) - f_{i+1}\| < \varepsilon$$

with $f_{d^n} = f_0$. Suppose we have chosen such projections $e_i, f_i$ for the same $n$. Since $K_0(\mathcal{O}_d) = \mathbf{Z}/(d-1)\mathbf{Z}$, we have that $[e_0] = 1 = [f_0]$ in $K_0(\mathcal{O}_d)$ and so obtain a partial isometry $w \in \mathcal{O}_d$ such that $w^*w = e_0$, $ww^* = f_0$. We find unitaries $v_1, v_2 \in \mathcal{O}_d$ such that $\mathrm{Ad}\,v_1\lambda(e_i) = e_{i+1}$, $\mathrm{Ad}\,v_2\,\mathrm{Ad}\,u\lambda(f_i) = f_{i+1}$, and $\|v_1 - 1\| \approx 0$, $\|v_2 - 1\| \approx 0$ (depending on $\varepsilon$). Let

$$z = w^*(L_{v_2u}R_{v_1^*}\lambda)^{d^n}(w)$$

where $R_{v_1^*}$ is the right multiplication by $v_1^*$ and $L_{v_2u}$ is the left multiplication by $v_2u$. Since $(L_{v_2u}R_{v_i^*}\lambda)^i(w)$ is a partial isometry with initial projection $e_i$ and final projection $f_i$, $z$ is a unitary in $e_0\mathcal{O}_de_0$. Since $K_1(\mathcal{O}_d) = 0$ and $\mathcal{O}_d$ has real rank zero, we find a sequence $z_0, z_1, \dots, z_{d^n-1}$ of unitaries in $e_0\mathcal{O}_de_0$ such that $z_0 = z$, $z_{d^n-1} = 1$,

$$\|z_i - z_{i+1}\| < 4/d^n\ .$$

Define a unitary $v$ by

$$v = \sum_{i=0}^{d^n-1} (L_{v_2u}R_{v_1^*}\lambda)^i(wz_i)$$

Then since

$$\begin{aligned} &v - (L_{v_2u}R_{v_1^*}\lambda)(v) \\ &\quad = \sum_{i=1}^{d^n-1} (L_{v_2u}R_{v_1^*}\lambda)^i(wz_i - wz_{i-1}) + wz_0 - (L_{v_2u}R_{v_1^*}\lambda)^{d^n}(w)\ , \end{aligned}$$

it follows that

$$\|v - L_{v_2u}R_{v_1^*}\lambda(v)\| < 4/d^n$$

or

$$\|v - u\lambda(v)\| \lesssim 4/d^n\ .$$

This completes the proof of (i).

Since $\mathcal{U}_a \supset \mathcal{U}_i$, $\mathcal{U}_a$ is dense. That $\mathcal{U}_a$ is a $G_\delta$ set follows from

$$\mathcal{U}_a = \bigcap_n \bigcap_j \bigcup_i \left\{ u \in \mathcal{U}(\mathcal{O}_d); \|\alpha_u(x_i) - x_j\| < \frac{1}{n} \right\}$$

where $\{x_i\}$ is a dense sequence in $\mathcal{O}_d$.

If $\mathcal{U}_a$ contains a non-empty open set, then it follows that $\mathcal{U}_a = \mathcal{U}(\mathcal{O}_d)$ or $\mathcal{U}_s = \emptyset$. Because for any unitaries $u, w$ of $\mathcal{O}_d$ we find a unitary $v$ such that $w\lambda(v) \approx vu$. (Apply the previous argument for the endomorphism $\operatorname{Ad} u \circ \lambda$ instead of $\lambda$ and the unitary $wu^*$.) Since $v\mathcal{U}_a\lambda(v^*) = \mathcal{U}_a$ for any unitary $v \in \mathcal{O}_d$, the above fact implies that $\mathcal{U}_a$ contains an arbitrary unitary. But we know that $\mathcal{U}_s \neq \emptyset$. For example if $u = \sum s_i s_j s_i^* s_j^*$, then $\alpha_u = \lambda$ and $\lambda(\mathcal{O}_d)' \simeq M_d$. Thus we obtain that $\mathcal{U}_s$ is dense. □

For a unit vector $\xi \in \mathbf{C}^d$ we have defined the Cuntz state $f_\xi$ of $\mathcal{O}_d$ by

$$f_\xi(s_{i_1} \dots s_{i_m} s_{j_n}^* \dots s_{j_1}^*) = \xi_{i_1} \dots \xi_{i_m} \overline{\xi_{j_n}} \dots \overline{\xi_{j_1}}$$

It follows that $f_\xi$ is a unique pure state of $\mathcal{O}_d$ satisfying

$$f_\xi\left( \sum_{i=1}^d \overline{\xi_i} s_i \right) = 1 \ .$$

Let $F$ be the linear span of $s_i s_j^*$, $i, j = 1, \dots, d$. Then $F$ is isomorphic to $M_d$ and each unitary $u$ in $F$ defines an automorphism $\alpha_u$ of $\mathcal{O}_d$. This group of automorphisms acts transitively on the compact set of Cuntz states.

We denote by $f_0$ the Cuntz state $f_\xi$ with $\xi = (1, 0, \dots, 0)$.

**Proposition 8.** *If $\varphi$ is a pure state of $\mathcal{O}_d$, there is a unital endomorphism $\alpha$ of $\mathcal{O}_d$ such that $\varphi \circ \alpha = f_0$, where $f_0$ is the Cuntz state defined above. Furthermore $\alpha$ may be chosen so that $\pi_\varphi \circ \alpha(\mathcal{O}_d)''$ contains the one-dimensional projection onto $\mathbf{C}\Omega_\varphi$.*

*Proof.* It suffices to show that if $\varphi$ is a pure state there is a unitary $u \in \mathcal{O}_d$ such that

$$\varphi(us_1) = 1 \ .$$

Since $\mathcal{O}_d$ has real rank zero, there is a decreasing sequence $(e_n)$ of projections in $\mathcal{O}_d$ such that $\varphi$ is the unique state satisfying $\varphi(e_n) = 1$ for $n = 1, 2, \dots$, i.e., $(e_n)$ converges to the support projection of $\varphi$ in $\mathcal{O}_d^{**}$. We may further assume that $[e_n] = 0$ in $K_0(\mathcal{O}_d)$.

Pick up a projection $e = e_n$ such that $\varphi(e) = 1$ and $e < 1$. Then $es_1^*$ is a partial isometry with initial projection $s_1 e s_1^*$ and final projection $e$. Let $w$ be a partial isometry such that $w^*w = 1 - s_1 e s_1^*$ and $ww^* = 1 - e$. Then $u = es_1^* + w$ is a unitary in $\mathcal{O}_d$ such that

$$us_1 e = (es_1^* + w)s_1 e = e \ .$$

Thus we have that $\varphi(us_1) = 1$.

To prove the last statement we shall modify $u$ so that $\varphi$ is the unique state satisfying

$$\varphi(us_1) = 1 \ .$$

We have chosen $e = e_n$. We let

$$h = \sum_{k=1}^\infty 2^{-k} e_{n+k} \ .$$

Then $h$ is self-adjoint with $0 \leq h \leq 1$ and $\varphi$ is the only state satisfying $\varphi(h) = 1$. Let

$$u_1 = e^{2\pi i h} u \ .$$

Then $u_1 s_1 e = e^{2\pi i h} e$ and the assertion follows. $\square$

## References


[BJ97] O. Bratteli and P.E.T. Jorgensen, Endomorphisms of $\mathcal{B}(\mathcal{H})$, II, *J. Funct. Anal.* **145** (1997), 323–373.

[BJP96] O. Bratteli, P.E.T. Jorgensen and G. Price, Endomorphisms of $\mathcal{B}(\mathcal{H})$, in W. Arveson et al., eds., "Quantization of Nonlinear Partial Differential Equations", Amer. Math. Soc. 1996.

[BJKW] O. Bratteli, P.E.T. Jorgensen, A. Kishimoto and R.F. Werner, Pure states on $\mathcal{O}_d$, J. Operator Theory, to appear.

[BK99] O. Bratteli and A. Kishimoto, Trace scaling automorphisms of certain stable AF algebras II, preprint 1999.

[BKRS93] O. Bratteli, A.K. Kishimoto, M. Rørdam and E. Størmer, The crossed product of a UHF algebra by a shift, *Ergodic Theory and Dyn. Sys.* **13** (1993), 615–626.

[Bra72] O. Bratteli, Inductive limits of finite dimensional C*-algebras, *Trans, Amer. Math. Soc.* **171** (1972), 195–234.

[Cun77] J. Cuntz, Simple C*-algebras generated by isometries, *Comm. Math. Phys.* **57** (1977), 173–185.

[Eva80] D.E. Evans, On $\mathcal{O}_n$, *Publ. RIMS. Kyoto Univ.* **16** (1980), 915–927.

[Kis95] A. Kishimoto, The Rohlin property for automorphisms of UHF algebras, *J. reine argew. Math.* **465** (1995), 183–196.

[KR86] R.V. Kadison and J.R. Ringrose, Fundamentals of the theory of operator algebras, Volume II, Academic Press 1986.

[KT78] A. Kishimoto and H. Takai, Some remarks on C*-dynamical systems with a compact abelian group, *Publ. Res. Inst. Math. Sci.* **14** (1978), 388–397.

[Pow67] R.T. Powers, Representations of uniformly hyperfinite algebras and their associated von Neumann rings, *Ann. of Math.* **86** (1967), 138–171.

[Rør93] M. Rørdam, Classification of inductive limits of Cuntz algebras, *J. reine angew. Math.* **440** (1993), 175–200.



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